MTH 201: Multivariable Calculus and Differential Equations

Problem Set 4: Green's, Stokes' and Gauss's Divergence Theorems

1 Properties of curl and divergence

1. If a scalar field $f(x, y, z)$ has continuous second partials, show that $\nabla \times \nabla f = 0$.

2. Let F_1 and F_2 be differentiable vector fields and let a and b be arbitrary real constants. Veirfy the following identitites.

   (a) $\nabla \cdot (aF_1 + bF_2) = a\nabla \cdot F_1 + b\nabla \cdot F_2$.

   (b) $\nabla \times (aF_1 + bF_2) = a(\nabla \times F_1) + b(\nabla \times F_2)$.

   (c) $\nabla \cdot (F_1 \times F_2) = F_2 \cdot (\nabla \times F_1) - F_1 \cdot (\nabla \times F_2)$.

   Let F be a differentiable vector field and let $g(x, y, z)$ be a differentiable scalar field. Verify the following identities.

   (a) $\nabla \cdot (gF) = g\nabla \cdot F + \nabla g \cdot F$

   (b) $\nabla \times (gF) = g\nabla \times F + \nabla g \times F$.

2 Green's Theorem

1. Use Green's Theorem to find the counteclockwise circulation and outward flux of the field F over C.

   (a) $F = (x^2 + 4y)i + (x + y^2)j$
   C : The square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$.

   (b) $F = (x + y)i - (x^2 + y^2)j$
   C : The triangle bounded by $y = 0$, $x = 1$, and $y = x$.

   (c) $F = \left(\tan^{-1} \frac{y}{x}\right) i + \ln(x^2 + y^2)j$
   C : The boundary of the region defined by the polar coordinate inequalities $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$.

   (d) $F = xyi + y^2j$
   C : The boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

2. Apply Green's Theorem to evaluate the following integrals.

   (a) $\displaystyle\int_c y^2\, dx + x^2\, dy$
   C : The triangle bounded by $x = 0$, $x + y = 1$, $y = 0$.

   (b) $\displaystyle\int_c (6y + x)\, dx + (y + 2x)\, dy$
   C : The circle $(x - 2)^2 + (y - 3)^2 = 4$.

3. If a simple closed curve C in the plane and the region R it encloses satisfy the hypthoses of the Green's Theorem, then show that the area of R is given by

$$\frac{1}{2} \int_C x\, dy - y\, dx.$$

4. Use the formula derived in the problem 3 to find the area of the region enclosed by C.

   (a) $r(t) = (a\cos t)i + (b\sin t)j$, $t \in [0, 2\pi]$.

   (b) $r(t) = (\cos^3 t)i + (\sin^3 t)j$, $t \in [0, 2\pi]$.

3 Stokes' Theorem

1. Use Stokes' Theorem to calculate the circulation of the field F around the curve C in the indicated direction.

   (a) $F = x^2 i + 2xj + z^2 k$
   C : The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above.

   (b) $F = (y^2 + z^2)i + (x^2 + z^2)j + (x^2 + y^2)k$
   C : The boundary of the traingle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above.

   (c) $F = x^2 y^3 i + j + zk$
   C : The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above.

2. Use Stokes' Theorem to calculate the flux of the curl of the field F across the surface S in the direction of the outward unit normal n, that is, $\iint_S (\nabla \times F) \cdot n\, d\sigma$.

   (a) $F = yi$
   S : The hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

   (b) $F = yi + xj + (x^2 + y^4)^{3/2} \sin e^{\sqrt{xyz}} k$
   S : The elliptical shell $4x^2 + 9y^2 + 36z^2 = 36$, $z \geq 0$.

   (c) $F = -yi + xj + x^2 k$
   S : The cylinder $x^2 + y^2 = a^2$, $z \in [0, h]$, together with its top, $x^2 + y^2 \leq a^2$, $z = h$.

   (d) $F = (y - z)i + (z - x)j + (x + z)k$
   S : $R(r, \theta) = (r\cos\theta)i + (r\sin\theta)j + (9 - r^2)k$, $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$.

   (e) $F = x^2 yi + 2y^2 zj + 3zk$
   S : $R(r\theta) = (r\cos\theta)i + (r\sin\theta)j + rk$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

   (f) $F = y^2 i + z^2 j + xk$
   S : $R(\phi, \theta) = (2\sin\phi\cos\theta)i + (2\sin\phi\sin\theta)j + (2\cos\phi)k$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$.

4 Gauss' Divergence Theorem

1. Use the Gauss' Divergence Theorem to find the outward flux of F across the boundary of the region D.

   (a) $F = \sqrt{x^2 + y^2 + z^2}(xi + yj + zk)$
   D : The spherical shell $1 \leq x^2 + y^2 + z^2 \leq 4$.

   (b) $F = x^2 + xzj + 3zk$
   D : The solid sphere $x^2 + y^2 + z^2 \leq 4$.

   (c) $F = x^2 i + y^2 j + z^2 k$
   D : The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$.

(d) $F = (y-x)i + (z-y)j + (y-x)k$
D : The region cut from the solid cylinder $x^2 + y^2 \leq 4$ by the planes $z = 0$ and $z = 1$.

(e) $F = yi + xyj - zk$
D : The region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$.

(f) $F = 2xzi - xyj - z^2k$
D : The wedge cut from the first octant by the plane $y + z = 4$ and the elliptical cylinder $4x^2 + y^2 = 16$.